Combinatorial Interpretation Of Some Rogers-Ramanujan Type Identities

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Abstract: In this paper we give the combinatorial interpretation of some analytical Rogers-Ramanujan type identities with the incorporation of a few results of Gordon and Andrews.

Keywords: Partitions, Combinatorial Interpretation, Rogers-Ramanujan type identities.

I. INTRODUCTION

THE ROGERS-RAMANUJAN IDENTITY

The following two identities, namely for \(|q| \leq 1\),

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(q^n;q)_n} \quad \text{where } n \neq 0, \pm 2 \pmod{5} \tag{1}
\]

\[
\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(q^n;q)_n} \quad \text{where } n \neq 0, \pm 1 \pmod{5} \tag{2}
\]

are the celebrated Rogers-Ramanujan Identity. These two identities, which have motivated extensive research over the past hundred years, were first discovered by L. J. Rogers in 1894 and these were again rediscovered independently by S. Ramanujan and I. Schur.

The Rogers-Ramanujan Identity has a long and interesting history. There are two aspects of these identities: one analytical and the other combinatorial. Our purpose in this paper is to give the combinatorial interpretation of some analytical identities of Rogers-Ramanujan type.

We begin by introducing some definitions, notations and then recalling some results:

II. DEFINITIONS AND NOTATIONS

For \(|q| \leq 1\), the q-shifted factorial is denoted by

\[(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k) \]

The multiple q-shifted factorial is

\[(a_1, a_2, \ldots, a_m)_n = (a_1; q)_n(a_2; q)_n \cdots (a_m; q)_n \]

And

\[(a_1, a_2, \ldots, a_m)_\infty = (a_1; q)_\infty(a_2; q)_\infty \cdots (a_m; q)_\infty. \]

DEFINITION

A partition of a positive integer \(n\) is a finite non-increasing sequence of positive integers \(\lambda_1, \lambda_2, \ldots, \lambda_r\) such that \(\sum_{i=1}^{r} \lambda_i = n\). The \(\lambda_i\)'s are called the parts of the partition. If \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)\) is a partition of a positive integer \(n\), then we write \(\lambda = (1 f_1, 2 f_2, 3 f_3, \ldots)\) where exactly \(f_i\) of the \(\lambda_j\) are equal to \(i\). We notice that \(\sum_{i=1}^{r} f_i i = n\).

The function \(p(n)\) denotes the number of partitions of \(n\).

III. FIRST WE REQUIRE THE FOLLOWING RESULTS

THEOREM (1): (BASIL GORDON)

Let \(1 \leq a \leq k\) be integers and \(B_{k,a}(n)\) denote the number of partitions of \(n\) of the form \(n = b_1 + b_2 + \cdots + b_k\), where \(b_i \geq b_{i+1} \text{ and } b_i - b_{i+k-1} \geq 2\) and \(1\) appears as a summand at most \((a-1)\) times. Let \(A_{k,a}(n)\) denote the number of partitions of \(n\) into parts not congruent to 0, \(\pm a\) mod \((2k+1)\). Then

\[A_{k,a}(n) = B_{k,a}(n), \forall n \in \mathbb{N}\.
\]

THEOREM (2): (BASIL GORDON)
Let $A_{d,k,i}(n)$ denote the number of partitions of $n$ into parts $\equiv 0, \pm di \mod (2dk+d)$. Let $B_{d,k,i}(n)$ denote the number of partitions of $n$ wherein $d$ appears as a part at most $i-1$ times,
- the total number of appearances of $dj$ and $dj+d$ (i.e. any two consecutive multiples of $d$) together is at most $k-1$, and
- nonmultiples of $d$ may appear as parts without restriction.

Then for $1 \leq i \leq k$,
$$A_{d,k,i}(n) = B_{d,k,i}(n), \quad \forall n \in \mathbb{N}. $$

**THEOREM (3): (ANDREWS)**

Let $A_{\lambda,k,a}(n)$ denote the number of partitions of $n$ of the form $n = b_1 + b_2 + \cdots + b_t$, where $b_i \geq b_{i+1}$, only parts divisible by $\lambda + 1$ may be repeated, $b_i - b_{i+k-1} \geq \lambda + 1$ (with strict inequality if $(\lambda + 1)|b_i$) and the total number of appearances of summands in the set $\{1, 2, \ldots, \lambda + 1\}$ is at most $a-1$. If $\lambda$ is even, let $A_{\lambda,k,a}(n)$ denote the number of partition of $n$ into parts such that no part $\equiv 0 \mod (\lambda + 1)$ may be repeated, and no part is congruent to $0$, $\pm \left(\frac{a-\lambda}{2}\right)(\lambda + 1)(\mod (2k - \lambda + 1)(\lambda + 1)).$

If $\lambda$ is odd, let $A_{\lambda,k,a}(n)$ denote the number of partitions of $n$ into parts such that no part $\equiv 0 \mod (\lambda + 1/2)$ may be repeated, and no part is congruent to $0$, $\pm (2a - \lambda) \left(\lambda + 1/2\right)(\mod (2k - \lambda + 1)(\lambda + 1)).$

Then provided $k \geq 2\lambda - 1$ and $k \geq a \geq \lambda$, we have $A_{\lambda,k,a}(n) = B_{\lambda,k,a}(n), \forall n \in \mathbb{N}.$


The analytical identity is
$$1 + \sum_{n=1}^{\infty} A_{2,1,1}(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}, \quad n \neq 0, \pm 2 \mod 6.$$ 

We use theorem (2). Let $d=2, i=1, k=1$ then $A_{1,1,1}(n)$ denote the number of partitions of $n$ into parts $\neq 0, \pm 2 \mod 6$. Also $B_{2,1,1}(n)$ denote the number of partitions of $n$ into parts wherein the parts are only non-multiples of 2. Then $A_{2,1,1}(n) = B_{2,1,1}(n), \quad \forall n \in \mathbb{N}.$

And we have for $|q| \leq 1.$
International Journal of Scientific Research and Engineering Studies (IJSRES)
Volume 1 Issue 5, November 2014
ISSN: 2349-8862

\[ \sum_{n \geq 0} \sum_{r \geq 0} \frac{(-1)^r q^{n^2+2n+ar(r-1)/2}}{(q;q)_{2n+2}(q^2;q^2)_r(q;q)_n-ar} = 1+\sum_{n=1}^{\infty} A_{10,3}(n) q^n \]

Where \( A_{10,3}(n) \) denotes the number of partitions of \( n \) into parts \( \not \equiv 0, \pm 3 \) (mod 21).

And \( B_{10,3}(n) \) denotes the number of partitions of \( n \) of the form \( n= b_1 + b_2 + \cdots b_r \), where \( b_i \geq b_{i+1} \) and \( b_i - b_{i+2} \geq 2 \) and 1 appears as a summand at most 2 times.


The analytical identity is

\[ \sum_{n \geq 1} \sum_{r \geq 0} \frac{(-1)^r q^{n^2+2r(r-1)/2}}{(q;q)_{2n-1}(q^2;q^2)_r(q;q)_n-ar} \]

\[ \prod_{n=1}^{\infty} \frac{1}{1-q^n} , \ n \not \equiv 0, \pm 6 \) (mod 21)

By using the theorem (1) for \( a=6, \ k=10 \), we have

\[ \sum_{n=1}^{\infty} A_{10,6}(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} , \ n \not \equiv 0, \pm 6 \) (mod 21)

\[ 1+ \sum_{n \geq 1} \sum_{r \geq 0} \frac{(-1)^r q^{n^2+2r(r-1)/2}}{(q;q)_{2n-1}(q^2;q^2)_r(q;q)_n-ar} = 1+\sum_{n=1}^{\infty} B_{10,3}(n) q^n \]

Where \( A_{10,6}(n) \) denotes the number of partitions of \( n \) into parts \( \not \equiv 0, \pm 6 \) (mod 21).

And \( B_{10,3}(n) \) denotes the number of partitions of \( n \) of the form \( n= b_1 + b_2 + \cdots b_r \), where \( b_i \geq b_{i+1} \) and \( b_i - b_{i+2} \geq 2 \) and 1 appears as a summand at most 5 times.

E. INTERPRETATION OF THE IDENTITY (3.50): ([7] P-255, MODULO 33)

The analytical identity is

Using the theorem (3) for \( \lambda=2, \ k=6, \ a=3 \), we have

\[ \sum_{n \geq 0} \sum_{r \geq 0} \frac{q^{n^2+2r^2+2(n-r)(q^2;q^2)_r(q;q)_n-ar}}{(q;q)_{2n+2}(q^2;q^2)_r(q;q)_n-ar} = 1+\sum_{n=1}^{\infty} A_{2,6,3}(n) q^n \]

\[ \prod_{n=1}^{\infty} \frac{1}{1-q^n} , \ n \not \equiv 0, \pm 6 \) (mod 33)

Here \( A_{2,6,3}(n) \) and \( B_{2,6,3}(n) \) are determined by theorem (3)

i.e \( A_{2,6,3}(n) \) is The number of partitions of \( n \) into parts \( \not \equiv 0, \pm 6 \) (mod 33) such that 3 and multiples of 3 may be repeated.

\( B_{2,6,3}(n) \) is the number of partitions of \( n \) of the form \( n= b_1 + b_2 + \cdots b_r \), where \( b_i \geq b_{i+1} \), only parts divisible by 3 may be repeated, \( b_i - b_{i+5} \geq 3 \) (with strict inequality if \( 3|b_i \)), and the parts 1, 2, 3 appearing at most once.

Then \( A_{2,6,3}(n) = B_{2,6,3}(n) , \ \forall n \in \mathbb{N} \).


The analytical identity is

\[ \sum_{n \geq 0} \sum_{r \geq 0} \frac{q^{n^2+2r^2+2(n-r)(q^2;q^2)_r(q;q)_n-ar}}{(q;q)_{2n+2}(q^2;q^2)_r(q;q)_n-ar} \]

\[ \prod_{n=1}^{\infty} \frac{1}{1-q^n} , \ n \not \equiv 0, \pm 6 \) (mod 33)

Using the theorem (3) for \( \lambda=2, \ k=6, \ a=3 \), we have

\[ 1+\sum_{n=1}^{\infty} A_{2,6,3}(n) q^n \]

\[ = \prod_{n=1}^{\infty} \frac{1}{1-q^n} , \ n \not \equiv 0, \pm 6 \) (mod 33)

\[ \sum_{n \geq 0} \sum_{r \geq 0} \frac{q^{n^2+2r^2+2(n-r)(q^2;q^2)_r(q;q)_n-ar}}{(q;q)_{2n+2}(q^2;q^2)_r(q;q)_n-ar} = 1+\sum_{n=1}^{\infty} B_{2,6,3}(n) q^n \]

Here \( A_{2,6,3}(n) \) and \( B_{2,6,3}(n) \) are determined by theorem (3)

i.e \( A_{2,6,3}(n) \) is The number of partitions of \( n \) into parts \( \not \equiv 0, \pm 6 \) (mod 33) such that 3 and multiples of 3 may be repeated.

\( B_{2,6,3}(n) \) is the number of partitions of \( n \) of the form \( n= b_1 + b_2 + \cdots b_r \), where \( b_i \geq b_{i+1} \), only parts divisible by 3 may be repeated, \( b_i - b_{i+5} \geq 3 \) (with strict inequality if \( 3|b_i \)), and the parts 1, 2, 3 appearing at most twice. Then

\[ A_{2,6,3}(n) = B_{2,6,3}(n) , \ \forall n \in \mathbb{N} \).

V. CONCLUSION

The combinatorial interpretation of most of the Rogers-Ramanujan Type Identities is possible. But the technique of
this paper cannot be used for the identities of Rogers-Ramanujan Type when the modulus arising in the infinite product of the form \(4k+2, k > 1\). In such case we have to take the concept of partitions with “n- copies of n” which is beyond the scope of this present paper.

REFERENCES